

Extensions and traces of functions of bounded variation on metric spaces

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Abstract

In the setting of a metric space equipped with a doubling measure and supporting a Poincaré inequality, and based on results by Björn and Shanmugalingam (2007, [7]), we show that functions of bounded variation can be extended from any bounded uniform domain to the whole space. Closely related to extensions is the concept of boundary traces, which have previously been studied by Hakkarainen et al. (2014, [12]). On spaces that satisfy a suitable locality condition for sets of finite perimeter, we establish some basic results for the traces of functions of bounded variation. Our analysis of traces also produces novel results on the behavior of functions of bounded variation in their jump sets.

1 Introduction

A classical Euclidean result on the extension of Sobolev functions and functions of bounded variation, abbreviated as BV functions, is that any bounded domain with a Lipschitz boundary allows such extensions, see e.g. [2, Proposition 3.21]. For Sobolev functions, this result was generalized to so-called (ε, δ) -domains by Jones [14]. On metric spaces, extension results for various classes of functions, such as Hajłasz-Sobolev functions and Hölder continuous functions, have been derived in e.g. [13] and [6]. One result on the extension of BV functions on metric spaces is given by Baldi and Montefalcone [4] who show, in essence, that if sets of finite perimeter can be extended from

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a domain, then so can general BV functions. However, a simple geometric condition ensuring the extendability of BV functions appears to be missing.

Björn and Shanmugalingam show in [7] that every *uniform domain* Ω is an extension domain for Newton-Sobolev functions $N^{1,p}(\Omega)$, with $p \geq 1$. On the other hand, BV functions on metric spaces are defined by relaxation with Newton-Sobolev functions, see [1] and [16]. Thus the extension result of [7] can be applied to the class BV in a fairly straightforward manner, as presented in this note.

On the other hand, the concepts of extensions and *boundary traces* are closely related. Classical treatments of boundary traces of BV functions can be found in e.g. [2, Chapter 3] and [10, Chapter 2], and a standard assumption is again a Lipschitz boundary. On the other hand, in the metric setting, results on boundary traces seem to be largely absent, with the exception of [12], where boundary traces of BV functions were defined on the boundaries of certain BV extension domains.

In this note, we present a different approach to traces, where more is assumed of the space but less of the domain. More precisely, we assume a certain locality condition that essentially states that any two sets of finite perimeter “look the same” near almost every point in which their measure theoretic boundaries intersect. Then we can prove the existence of *interior traces* of BV functions on the measure theoretic boundary of any set of finite perimeter, and also prove the existence of boundary traces on the measure theoretic boundary of any extension domain.

In [15], pointwise properties of BV functions on metric spaces were studied, and in particular a Lebesgue point theorem for BV functions outside their jump sets was given. Since the super-level sets of a BV function are sets of finite perimeter, we are able to apply our analysis of traces to prove novel results on the behavior of a BV function in its jump set, extending classical results to metric spaces and strengthening results found in [15].

2 Preliminaries

In this section we introduce the necessary definitions and assumptions.

In this paper, (X, d, μ) is a complete metric space equipped with a Borel regular outer measure μ . The measure is assumed to be doubling, meaning that there exists a constant $c_d > 0$ such that

$$0 < \mu(B(x, 2r)) \leq c_d \mu(B(x, r)) < \infty$$

for every ball $B = B(x, r)$ with center $x \in X$ and radius $r > 0$. This implies

that

$$\frac{\mu(B(y, r))}{\mu(B(x, R))} \geq C \left(\frac{r}{R} \right)^Q \quad (2.1)$$

for every $0 < r \leq R$ and $y \in B(x, R)$, and some $Q > 1$ and $C > 0$ that only depend on c_d . In general, C will denote a positive constant whose value is not necessarily the same at each occurrence.

We recall that a complete metric space endowed with a doubling measure is proper, that is, closed and bounded sets are compact. Since X is proper, for any open set $\Omega \subset X$ we define e.g. $\text{Lip}_{\text{loc}}(\Omega)$ as the space of functions that are Lipschitz in every $\Omega' \Subset \Omega$. Here $\Omega' \Subset \Omega$ means that Ω' is open and that $\overline{\Omega'}$ is a compact subset of Ω .

For any set $A \subset X$ and $0 < R < \infty$, the restricted spherical Hausdorff content of codimension 1 is defined as

$$\mathcal{H}_R(A) := \inf \left\{ \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} : A \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), r_i \leq R \right\}.$$

The Hausdorff measure of codimension 1 of a set $A \subset X$ is

$$\mathcal{H}(A) := \lim_{R \rightarrow 0} \mathcal{H}_R(A).$$

The (topological) boundary ∂E of a set $E \subset X$ is defined as usual. The measure theoretic boundary $\partial^* E$ is defined as the set of points $x \in X$ in which both E and its complement have positive upper density, i.e.

$$\limsup_{r \rightarrow 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} > 0 \quad \text{and} \quad \limsup_{r \rightarrow 0} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} > 0.$$

A curve is a rectifiable continuous mapping from a compact interval to X , and is usually denoted by the symbol γ . The length of a curve γ is denoted by ℓ_γ . We will assume every curve to be parametrized by arc-length, which can always be done (see e.g. [11, Theorem 3.2]).

A nonnegative Borel function g on X is an upper gradient of an extended real-valued function u on X if for all curves γ on X , we have

$$|u(x) - u(y)| \leq \int_\gamma g \, ds$$

whenever both $u(x)$ and $u(y)$ are finite, and $\int_\gamma g \, ds = \infty$ otherwise. Here x and y are the end points of γ .

We consider the following norm

$$\|u\|_{N^{1,1}(X)} := \|u\|_{L^1(X)} + \inf_g \|g\|_{L^1(X)},$$

with the infimum taken over all upper gradients g of u . The Newton-Sobolev, or Newtonian space is defined as

$$N^{1,1}(X) := \{u : \|u\|_{N^{1,1}(X)} < \infty\} / \sim,$$

where the equivalence relation \sim is given by $u \sim v$ if and only if

$$\|u - v\|_{N^{1,1}(X)} = 0.$$

Similarly, we can define $N^{1,1}(\Omega)$ for an open set $\Omega \subset X$. For more on Newtonian spaces, we refer to [5].

Next we recall the definition and basic properties of functions of bounded variation on metric spaces, see [16]. For $u \in L^1_{\text{loc}}(X)$, we define the total variation of u as

$$\|Du\|(X) := \inf \left\{ \liminf_{i \rightarrow \infty} \int_X g_{u_i} d\mu : u_i \in \text{Lip}_{\text{loc}}(X), u_i \rightarrow u \text{ in } L^1_{\text{loc}}(X) \right\},$$

where g_{u_i} is an upper gradient of u_i . We say that a function $u \in L^1(X)$ is of bounded variation, and denote $u \in \text{BV}(X)$, if $\|Du\|(X) < \infty$. Moreover, a μ -measurable set $E \subset X$ is said to be of finite perimeter if $\|D\chi_E\|(X) < \infty$. By replacing X with an open set $\Omega \subset X$ in the definition of the total variation, we can define $\|Du\|(\Omega)$. The BV norm is given by

$$\|u\|_{\text{BV}(\Omega)} := \|u\|_{L^1(\Omega)} + \|Du\|(\Omega).$$

For an arbitrary set $A \subset X$, we define

$$\|Du\|(A) := \inf \{ \|Du\|(\Omega) : \Omega \supset A, \Omega \subset X \text{ is open} \}.$$

If $u \in \text{BV}(\Omega)$, $\|Du\|(\cdot)$ Radon measure of finite mass on Ω [16, Theorem 3.4].

We also denote the perimeter of E in Ω by

$$P(E, \Omega) := \|D\chi_E\|(\Omega).$$

We have the following coarea formula given by Miranda in [16, Proposition 4.2]: if $F \subset X$ is a Borel set and $u \in \text{BV}(X)$, we have

$$\|Du\|(F) = \int_{-\infty}^{\infty} P(\{u > t\}, F) dt. \quad (2.2)$$

We always assume that X supports a $(1, 1)$ -Poincaré inequality, meaning that for some constants $c_P > 0$ and $\lambda \geq 1$, for every ball $B(x, r)$, for every locally integrable function u , and for every upper gradient g of u , we have

$$\int_{B(x, r)} |u - u_{B(x, r)}| d\mu \leq c_P r \int_{B(x, \lambda r)} g d\mu,$$

where

$$u_{B(x,r)} := \fint_{B(x,r)} u \, d\mu := \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u \, d\mu.$$

The $(1, 1)$ -Poincaré inequality implies the so-called Sobolev-Poincaré inequality, see e.g. [5, Theorem 4.21], and by approximation with get the following Sobolev-Poincaré inequality for BV functions. There exists $C > 0$, depending only on the doubling constant and the constants in the Poincaré inequality, such that for every ball $B(x, r)$ and every $u \in L^1_{\text{loc}}(X)$, we have

$$\left(\fint_{B(x,r)} |u - u_{B(x,r)}|^{Q/(Q-1)} \, d\mu \right)^{(Q-1)/Q} \leq Cr \frac{\|Du\|(B(x, 2\lambda r))}{\mu(B(x, 2\lambda r))}.$$

Recall the definition of the number Q from (2.1). Moreover, for functions $u \in L^1_{\text{loc}}(X)$ with approximate limit 0 at x , i.e. $u^\wedge(x) = u^\vee(x) = 0$, we have

$$\limsup_{r \rightarrow 0} \left(\fint_{B(x,r)} |u|^{Q/(Q-1)} \, d\mu \right)^{(Q-1)/Q} \leq C \limsup_{r \rightarrow 0} r \frac{\|Du\|(B(x, 2\lambda r))}{\mu(B(x, 2\lambda r))}, \quad (2.3)$$

see [15, Lemma 3.1].

Given a set of locally finite perimeter $E \subset X$, for \mathcal{H} -a.e. $x \in \partial^* E$ we have

$$\gamma \leq \liminf_{r \rightarrow 0} \frac{\mu(E \cap B(x, r))}{\mu(B(x, r))} \leq \limsup_{r \rightarrow 0} \frac{\mu(E \cap B(x, r))}{\mu(B(x, r))} \leq 1 - \gamma, \quad (2.4)$$

where $\gamma \in (0, 1/2]$ only depends on the doubling constant and the constants in the Poincaré inequality [1, Theorem 5.4]. For a Borel set $F \subset X$ and a set of finite perimeter $E \subset X$, we know that

$$\|D\chi_E\|(F) = \int_{\partial^* E \cap F} \theta_E \, d\mathcal{H}, \quad (2.5)$$

where $\theta_E : X \mapsto [\alpha, c_d]$, with $\alpha = \alpha(c_d, c_P, \lambda) > 0$, see [1, Theorem 5.3] and [3, Theorem 4.6].

The jump set of $u \in \text{BV}(X)$ is defined as

$$S_u := \{x \in X : u^\wedge(x) < u^\vee(x)\},$$

where u^\wedge and u^\vee are the lower and upper approximate limits of u defined as

$$u^\wedge(x) := \sup \left\{ t \in \overline{\mathbb{R}} : \lim_{r \rightarrow 0} \frac{\mu(B(x, r) \cap \{u < t\})}{\mu(B(x, r))} = 0 \right\}$$

and

$$u^\vee(x) := \inf \left\{ t \in \overline{\mathbb{R}} : \lim_{r \rightarrow 0} \frac{\mu(B(x, r) \cap \{u > t\})}{\mu(B(x, r))} = 0 \right\}.$$

Outside the jump set, i.e. in $X \setminus S_u$, \mathcal{H} -almost every point is a Lebesgue point of u [15, Theorem 3.5], and we denote the Lebesgue limit at x by $\tilde{u}(x)$.

The following decomposition result holds for the variation measure of a BV function. Given an open set $\Omega^* \subset X$, a function $u \in \text{BV}(\Omega^*)$, and a Borel set $A \subset \Omega^*$ that is σ -finite with respect to \mathcal{H} , we have

$$\|Du\|(\Omega^*) = \|Du\|(\Omega^* \setminus A) + \int_A \int_{u^\wedge(x)}^{u^\vee(x)} \theta_{\{u>t\}}(x) dt d\mathcal{H}(x), \quad (2.6)$$

see [3, Theorem 5.3].

A domain $\Omega \subset X$ is said to be A -uniform, with constant $A \geq 1$, if for every $x, y \in \Omega$ there exists a curve γ in Ω connecting x and y such that $\ell_\gamma \leq A d(x, y)$, and for all $t \in [0, \ell_\gamma]$, we have

$$\text{dist}(\gamma(t), X \setminus \Omega) \geq A^{-1} \min\{t, \ell_\gamma - t\}.$$

We say that a μ -measurable set Ω satisfies the weak measure density condition if for \mathcal{H} -a.e. $x \in \partial\Omega$,

$$\liminf_{r \rightarrow 0} \frac{\mu(B(x, r) \cap \Omega)}{\mu(B(x, r))} > 0. \quad (2.7)$$

In particular, this is true for any uniform domain [7].

3 The extension result

In this section we present the extension result for BV functions. For any $t > 0$ and any set $\Omega \subset X$, we define

$$\Omega^t := \{x \in X : \text{dist}(x, \Omega) < t\}$$

and

$$\Omega_t := \{x \in \Omega : \text{dist}(x, X \setminus \Omega) > t\}.$$

Theorem 3.1. *Let $A \geq 1$, let $\Omega \subset X$ be a bounded A -uniform domain, and let $u \in \text{BV}(\Omega)$. Let $T \in (0, \text{diam}(\Omega))$. Then there is an extension $Eu \in \text{BV}(X)$ such that $Eu|_\Omega = u$, $\text{supp}(Eu) \subset \Omega^T$,*

$$\|Eu\|_{\text{BV}(X)} \leq C \|u\|_{\text{BV}(\Omega)},$$

and $\|D(Eu)\|(\partial\Omega) = 0$. The constant C depends only on the doubling constant, the constants in the Poincaré inequality, A , and T .

Remark 3.2. According to [7, Theorem 5.6], on which the above result is based, we can in fact replace the uniformity assumption by the following assumptions: $\mu(\partial\Omega) = 0$, Lipschitz functions are dense in $N^{1,1}(\Omega)$, $\overline{\Omega}$ satisfies a *corkscrew condition*, and the weighted measure $\text{dist}(x, X \setminus \Omega)^\alpha d\mu(x)$ with some $\alpha > 0$ supports a $(1,1)$ -Poincaré inequality on $\overline{\Omega}$. For the precise definitions of these concepts, see [7]. In particular, all of these conditions follow from the uniformity assumption.

Proof. In the proof of [7, Theorem 5.6], the authors first construct an extension operator $F : L^1(\Omega) \rightarrow L^1(X)$ that maps $\text{Lip}(\Omega)$ to $\text{Lip}(\Omega^T)$. An analysis of the proof reveals that F satisfies for any $t \in (0, T)$

$$\int_{\Omega^t \setminus \Omega} |Fv| d\mu \leq C \int_{\Omega \setminus \Omega_{\alpha t}} |v| d\mu \quad \text{and} \quad \int_{\Omega^t \setminus \Omega} g_{Fv} d\mu \leq C \int_{\Omega \setminus \Omega_{\alpha t}} g_v d\mu,$$

where the first inequality holds for any $v \in L^1(\Omega)$, and the second for any $v \in \text{Lip}(\Omega)$, and the constants $C, \alpha > 0$ depend only on c_d, c_P, λ, A , and T . Moreover, g_{Fv} and g_v are upper gradients of Fv and v . Then one can define a cutoff function

$$\eta(x) := \max\{0, \min\{1, 2 - \text{dist}(x, \Omega)/(T/4)\}\}$$

and set $Ev := \eta Fv$. Now, if $v \in \text{Lip}(\Omega)$, then $g_{Ev} := g_{Fv}\eta + g_\eta|Fv|$ is an upper gradient of Ev according to the Leibniz rule (see [5, Theorem 2.15]), and we get the estimates for any $t \in (0, T)$:

$$\int_{\Omega^t \setminus \Omega} |Ev| d\mu \leq C \int_{\Omega \setminus \Omega_{\alpha t}} |v| d\mu \quad \text{for } v \in L^1(\Omega), \quad (3.1)$$

and

$$\int_{\Omega^t \setminus \Omega} g_{Ev} d\mu \leq C \int_{\Omega \setminus \Omega_{\alpha t}} g_v d\mu + \frac{C}{T} \int_{\Omega \setminus \Omega_{\alpha t}} |v| d\mu \quad \text{for } v \in \text{Lip}(\Omega). \quad (3.2)$$

Of course, in the second inequality the number T can then be absorbed into the constant C . Now, take a sequence $u_i \in \text{Lip}_{\text{loc}}(\Omega)$ such that $u_i \rightarrow u$ in $L^1_{\text{loc}}(\Omega)$ and

$$\int_{\Omega} g_{u_i} d\mu \rightarrow \|Du\|(\Omega) \quad (3.3)$$

as $i \rightarrow \infty$. If u is bounded, we can truncate the functions u_i , if necessary, to obtain $u_i \rightarrow u$ in $L^1(\Omega)$. If u is unbounded, for the truncated functions we have

$$u^k := \min\{k, \max\{-k, u\}\} \rightarrow u \quad \text{in } L^1(\Omega) \quad \text{as } k \rightarrow \infty,$$

and $\|Du^k\|(\Omega) \rightarrow \|Du\|(\Omega)$ by the lower semicontinuity of the variation measure. Thus in any case, we can assume that $u_i \rightarrow u$ in $L^1(\Omega)$. As noted in Remark 3.2, Lipschitz functions are dense in $N^{1,1}(\Omega)$. Thus we can also assume that $u_i \in \text{Lip}(\Omega)$ for all $i \in \mathbb{N}$. We can extend each function u_i to $Eu_i \in \text{Lip}_c(\Omega^T)$, such that by (3.2), we have

$$\int_{\Omega^T} g_{Eu_i} d\mu \leq C \int_{\Omega} g_{u_i} d\mu + C \int_{\Omega} |u_i| d\mu$$

for every $i \in \mathbb{N}$. Then

$$\liminf_{i \rightarrow \infty} \int_{\Omega^T} g_{Eu_i} d\mu \leq C \|u\|_{\text{BV}(\Omega)}.$$

We can also extend u to $Eu \in L^1(X)$, and by applying (3.1) to $|Eu_i - Eu|$, we get

$$\int_X |Eu_i - Eu| d\mu \leq C \int_{\Omega} |u_i - u| d\mu \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

Thus we have by (3.1) and by the definition of the variation measure,

$$\int_X |Eu| d\mu \leq C \int_{\Omega} |u| d\mu \quad \text{and} \quad \|D(Eu)\|(X) \leq C \|u\|_{\text{BV}(\Omega)}.$$

This shows that $u \in \text{BV}(X)$ and $\|Eu\|_{\text{BV}(X)} \leq C \|u\|_{\text{BV}(\Omega)}$, with $C = C(c_d, c_P, \lambda, A, T)$.

Finally let us show that $\|D(Eu)\|(\partial\Omega) = 0$. By (3.2), we get for the sequence $u_i \in \text{Lip}(\Omega)$ and for any $t \in (0, T)$ that

$$\begin{aligned} \limsup_{i \rightarrow \infty} \int_{\Omega^t \setminus \overline{\Omega_{\alpha t}}} g_{Eu_i} d\mu &\leq C \limsup_{i \rightarrow \infty} \int_{\Omega \setminus \Omega_{\alpha t}} g_{u_i} d\mu + C \limsup_{i \rightarrow \infty} \int_{\Omega \setminus \Omega_{\alpha t}} |u_i| d\mu \\ &\leq C \|Du\|(\Omega \setminus \Omega_{\alpha t}) + C \int_{\Omega \setminus \Omega_{\alpha t}} |u| d\mu. \end{aligned}$$

The last inequality follows from the definition of the variation measure, since we have $u_i \rightarrow u$ in $L^1(\Omega)$ and (3.3). By using this definition again, and recalling that $Eu_i \rightarrow Eu$ in $L^1(X)$, we get

$$\|D(Eu)\|(\partial\Omega) \leq \|D(Eu)\|(\Omega^t \setminus \overline{\Omega_{\alpha t}}) \leq \|Du\|(\Omega \setminus \Omega_{\alpha t}) + C \int_{\Omega \setminus \Omega_{\alpha t}} |u| d\mu.$$

By letting $t \rightarrow 0$, we get $\|D(Eu)\|(\partial\Omega) = 0$. □

We give the following definition.

Definition 3.3. An open set $\Omega \subset X$ that satisfies the conclusions of Theorem 3.1 is a *strong BV extension domain*. Additionally, we say that an open set $\Omega \subset X$ is a BV extension domain with a constant $c_\Omega > 0$ if for every $u \in \text{BV}(\Omega)$, there is an extension $Eu \in \text{BV}(X)$ with $Eu|_\Omega = u$ and $\|Eu\|_{\text{BV}(X)} \leq c_\Omega \|u\|_{\text{BV}(\Omega)}$.

Thus the difference between a BV extension domain and a strong BV extension domain is that for a BV extension domain, we do not require $\|D(Eu)\|(\partial\Omega) = 0$.

4 Traces of BV functions

Closely related to extensions is the concept of boundary traces. We give the following definition.

Definition 4.1. For a μ -measurable set $\Omega \subset X$ and a μ -measurable function u on Ω , a function $T_\Omega u$ defined on $\partial\Omega$ is a boundary trace of u if for \mathcal{H} -a.e. $x \in \partial\Omega$, we have

$$\lim_{r \rightarrow 0} \int_{B(x,r) \cap \Omega} |u - T_\Omega u(x)| d\mu = 0.$$

For classical results on boundary traces of BV functions in the Euclidean setting, see e.g. [2, Chapter 3] or [10, Chapter 2]. As regards the metric setting, in [12, Theorem 5.7] it was shown that if Ω is a strong BV extension domain and satisfies the weak measure density condition (2.7), then the boundary trace $T_\Omega u$ exists, that is, $T_\Omega u(x)$ is well-defined for \mathcal{H} -a.e. $x \in \partial\Omega$. The proof is based on the fact that since $\|D(Eu)\|(\partial\Omega) = 0$, the boundary $\partial\Omega$ and the jump set S_{Eu} of the extension Eu intersect only in a set of \mathcal{H} -measure zero, due to (2.6).

Next we consider a somewhat different approach to traces, which requires less to be assumed of the set Ω . The following lemma will be useful, and the proof will be similar to the one used in the Euclidean case in [8].

Lemma 4.2. Let $u \in \text{BV}(X)$ and let $\Omega \subset X$ be a μ -measurable set. Consider points $x \in S_u$ for which

$$\liminf_{r \rightarrow 0} \frac{\mu(B(x,r) \cap \Omega)}{\mu(B(x,r))} \geq c, \tag{4.1}$$

where $c > 0$ is a constant, and

$$\lim_{r \rightarrow 0} \frac{\mu(B(x,r) \cap \Omega \setminus \{u > t\})}{\mu(B(x,r) \cap \Omega)} = 0 \tag{4.2}$$

for every $t < u^\vee(x)$. For \mathcal{H} -a.e. such point, we have

$$\lim_{r \rightarrow 0} \int_{B(x,r) \cap \Omega} |u - u^\vee(x)|^{Q/(Q-1)} d\mu = 0,$$

where the number $Q > 1$ was defined in (2.1). The corresponding result for $u^\wedge(x)$ can be formulated similarly.

Proof. Let $\varepsilon > 0$, and let $x \in X$ satisfy all the assumptions of the lemma. We can also assume that $-\infty < u^\wedge(x) < u^\vee(x) < \infty$, as this holds for \mathcal{H} -a.e. $x \in S_u$ [15, Lemma 3.2]. Given any $r > 0$, we calculate

$$\begin{aligned} & \int_{B(x,r) \cap \Omega} |u - u^\vee(x)|^{Q/(Q-1)} d\mu \\ &= \frac{1}{\mu(B(x,r) \cap \Omega)} \int_{B(x,r) \cap \Omega \cap \{u^\vee(x) - \varepsilon < u < u^\vee(x) + \varepsilon\}} |u - u^\vee(x)|^{Q/(Q-1)} d\mu \\ & \quad + \frac{1}{\mu(B(x,r) \cap \Omega)} \int_{B(x,r) \cap \Omega \setminus \{u > u^\vee(x) - \varepsilon\}} |u - u^\vee(x)|^{Q/(Q-1)} d\mu \\ & \quad + \frac{1}{\mu(B(x,r) \cap \Omega)} \int_{B(x,r) \cap \Omega \cap \{u \geq u^\vee(x) + \varepsilon\}} |u - u^\vee(x)|^{Q/(Q-1)} d\mu. \end{aligned} \tag{4.3}$$

The first term on the right-hand side is clearly at most $\varepsilon^{Q/(Q-1)}$. Let $M > 0$ with $-M < u^\vee(x) - \varepsilon$. The second term can be estimated as follows:

$$\begin{aligned} & \frac{1}{\mu(B(x,r) \cap \Omega)} \int_{B(x,r) \cap \Omega \setminus \{u > u^\vee(x) - \varepsilon\}} |u - u^\vee(x)|^{Q/(Q-1)} d\mu \\ & \leq | -M - u^\vee(x) |^{Q/(Q-1)} \frac{\mu(B(x,r) \cap \Omega \setminus \{u > u^\vee(x) - \varepsilon\})}{\mu(B(x,r) \cap \Omega)} \\ & \quad + \frac{1}{\mu(B(x,r) \cap \Omega)} \int_{B(x,r) \cap \{u < -M\}} |u - u^\vee(x)|^{Q/(Q-1)} d\mu. \end{aligned}$$

By (4.2), the first term on the right-hand side goes to zero as $r \rightarrow 0$. The third term of (4.3) can be estimated similarly, using the definition of the approximate upper limit, provided we also require that $M > u^\vee(x) + \varepsilon$. In total, we get

$$\begin{aligned} & \limsup_{r \rightarrow 0} \int_{B(x,r) \cap \Omega} |u - u^\vee(x)|^{Q/(Q-1)} d\mu \\ & \leq \varepsilon^{Q/(Q-1)} + \limsup_{r \rightarrow 0} \frac{1}{\mu(B(x,r) \cap \Omega)} \int_{B(x,r) \cap \{|u| > M\}} |u - u^\vee(x)|^{Q/(Q-1)} d\mu. \end{aligned}$$

Here we have by (4.1)

$$\begin{aligned}
& \limsup_{r \rightarrow 0} \frac{1}{\mu(B(x, r) \cap \Omega)} \int_{B(x, r) \cap \{u > M\}} |u - u^\vee(x)|^{Q/(Q-1)} d\mu \\
& \leq 2^{Q/(Q-1)} \limsup_{r \rightarrow 0} \frac{1}{c\mu(B(x, r))} \int_{B(x, r)} (u - M)_+^{Q/(Q-1)} d\mu \\
& \quad + 2^{Q/(Q-1)} |M - u^\vee(x)|^{Q/(Q-1)} \limsup_{r \rightarrow 0} \frac{\mu(B(x, r) \cap \{u > M\})}{c\mu(B(x, r))} \\
& \leq C \limsup_{r \rightarrow 0} \left(r \frac{\|D(u - M)_+\|(B(x, r))}{\mu(B(x, r))} \right)^{Q/(Q-1)},
\end{aligned}$$

where the last inequality follows from the fact that $M > u^\vee(x)$, as well as (2.3). Moreover, $C = C(c_d, c_P, \lambda, c)$. An analogous estimate holds for the set $\{u < -M\}$, provided that we also have $-M < u^\wedge(x)$, and then in total we get

$$\begin{aligned}
& \limsup_{r \rightarrow 0} \left(\int_{B(x, r) \cap \Omega} |u - u^\vee(x)|^{Q/(Q-1)} d\mu \right)^{(Q-1)/Q} \\
& \leq \varepsilon + C \limsup_{r \rightarrow 0} r \frac{\|D(u - M)_+\|(B(x, r))}{\mu(B(x, r))} \\
& \quad + C \limsup_{r \rightarrow 0} r \frac{\|D(u + M)_-\|(B(x, r))}{\mu(B(x, r))}.
\end{aligned}$$

Since the number M can be chosen to be arbitrarily large, it is straightforward to show that the right-hand side of the above inequality is smaller than $C\varepsilon$ outside a set of arbitrarily small \mathcal{H} -measure (see e.g. the proof of [15, Theorem 3.5]). Since $\varepsilon > 0$ was arbitrary, we have the result. \square

Before considering traces, we prove the following result which is in close relation with the concept of traces. The theorem strengthens [15, Theorem 1.1].

Theorem 4.3. *Let $u \in \text{BV}(X)$. Then for \mathcal{H} -a.e. $x \in S_u$, there exist $t_1, t_2 \in (u^\wedge(x), u^\vee(x))$ such that*

$$\lim_{r \rightarrow 0} \int_{B(x, r) \cap \{u > t_2\}} |u - u^\vee(x)|^{Q/(Q-1)} d\mu = 0$$

and

$$\lim_{r \rightarrow 0} \int_{B(x, r) \cap \{u < t_1\}} |u - u^\wedge(x)|^{Q/(Q-1)} d\mu = 0.$$

Proof. In this proof we denote, for brevity, the super-level sets of u by $E_t := \{u > t\}$, $t \in \mathbb{R}$. By the coarea formula (2.2), there is a countable dense set $T \subset \mathbb{R}$ such that for every $t \in T$, the set E_t is of finite perimeter. Let

$$N := \bigcup_{t \in T} \{x \in \partial^* E_t : (2.4) \text{ does not hold at } x \text{ with } E \hookrightarrow E_t\} \quad (4.4)$$

and

$$\tilde{N} := \bigcup_{s, t \in T} \{x \in \partial^*(E_s \setminus E_t) : (2.4) \text{ does not hold at } x \text{ with } E \hookrightarrow E_s \setminus E_t\}. \quad (4.5)$$

Since the sets $E_s \setminus E_t$, $s, t \in T$, are also of finite perimeter by [16, Proposition 4.7], we have $\mathcal{H}(N \cup \tilde{N}) = 0$. From the definitions of the lower and upper approximate limits it follows that whenever $x \in S_u$, it is true that $x \in \partial^* E_t$ for every $t \in (u^\wedge(x), u^\vee(x))$. Now, at a point $x \in S_u \setminus (N \cup \tilde{N})$, the number t_2 can be chosen as follows. If

$$\limsup_{r \rightarrow 0} \frac{\mu(B(x, r) \cap E_s \setminus E_t)}{\mu(B(x, r))} > 0 \quad (4.6)$$

for $s, t \in T \cap (u^\wedge(x), u^\vee(x))$, then we have $x \in \partial^*(E_s \setminus E_t)$, and since $x \notin \tilde{N}$, we actually have

$$\liminf_{r \rightarrow 0} \frac{\mu(B(x, r) \cap E_s \setminus E_t)}{\mu(B(x, r))} \geq \gamma.$$

Here $\gamma > 0$ is a constant. Thus, if (4.6) holds for all consecutive numbers in an increasing sequence $s < t < \dots \in T \cap (u^\wedge(x), u^\vee(x))$, the sequence must be finite, and so we have

$$\limsup_{r \rightarrow 0} \frac{\mu(B(x, r) \cap E_{t_2} \setminus E_t)}{\mu(B(x, r))} = 0$$

for some $t_2 \in (u^\wedge(x), u^\vee(x)) \cap T$ and all $t \in (t_2, u^\vee(x))$.

Finally, since the set T is countable, the union of the exceptional sets of Lemma 4.2, with $\Omega \hookrightarrow E_s$ and $s \in T$, has \mathcal{H} -measure zero. Thus we can assume that x is outside this set, and then Lemma 4.2 gives the result for the set $E_{t_2} = \{u > t_2\}$. The proof for the number t_1 and the set $\{u < t_1\}$ is analogous. \square

Now we proceed to consider traces. First we present an additional assumption on the space X . Following [3, Definition 6.1], we say that a space satisfies the *locality condition* if, given any two sets of locally finite perimeter $E_1, E_2 \subset X$, we have $\theta_{E_1}(x) = \theta_{E_2}(x)$ for \mathcal{H} -a.e. $x \in \partial^* E_1 \cap \partial^* E_2$ — recall

the definition of θ_E from (2.5). The above could as well be formulated with the additional assumption $E_1 \subset E_2$, see the discussion following Definition 5.9 in [12].

Here we give the following stronger condition.

Definition 4.4. The space X satisfies the *strong locality condition* if, given any two sets of locally finite perimeter $E_1 \subset E_2 \subset X$, we have for \mathcal{H} -a.e. $x \in \partial^* E_1 \cap \partial^* E_2$

$$\lim_{r \rightarrow 0} \frac{\mu((E_2 \setminus E_1) \cap B(x, r))}{\mu(B(x, r))} = 0. \quad (4.7)$$

Strong locality is indeed stronger than locality, as we will soon see.

Condition (4.7) is, in particular, satisfied if for any sets of locally finite perimeter $E_1 \subset E_2 \subset X$, the sets E_1 and E_2 have the same density at \mathcal{H} -a.e. $x \in \partial^* E_1 \cap \partial^* E_2$. Furthermore, the assumption $E_1 \subset E_2$ is again essentially unnecessary, as the following lemma demonstrates.

Lemma 4.5. *Assume that X satisfies the strong locality condition. Let $E_1, E_2 \subset X$ be sets of locally finite perimeter. Then for \mathcal{H} -a.e. $x \in \partial^* E_1 \cap \partial^* E_2$, we have either*

$$\lim_{r \rightarrow 0} \frac{\mu((E_1 \triangle E_2) \cap B(x, r))}{\mu(B(x, r))} = 0, \quad (4.8)$$

or the same with the substitution $E_2 \hookrightarrow E_2^c$ (complement of E_2). Here \triangle is the symmetric difference.

Proof. First we note that E_2^c , $E_1 \cap E_2$, and $E_1 \cap E_2^c$ are also sets of locally finite perimeter, see [16, Proposition 4.7]. Take a point $x \in \partial^* E_1 \cap \partial^* E_2$. For any sets $A_1 \subset A_2 \subset X$, let N_{A_1, A_2} be the set of points $x \in \partial^* A_1 \cap \partial^* A_2$ for which

$$\limsup_{r \rightarrow 0} \frac{\mu((A_2 \setminus A_1) \cap B(x, r))}{\mu(B(x, r))} > 0.$$

By (4.7), excluding a \mathcal{H} -negligible set we can assume that

$$x \notin N_{E_1 \cap E_2, E_1} \cup N_{E_1 \cap E_2, E_2} \cup N_{E_1 \cap E_2^c, E_1} \cup N_{E_1 \cap E_2^c, E_2^c}.$$

Since $x \in \partial^* E_1 \cap \partial^* E_2$, by the definition of the measure theoretic boundary we have either $x \in \partial^*(E_1 \cap E_2)$ or $x \in \partial^*(E_1 \cap E_2^c)$. Assume the former. Using (4.7), we can calculate

$$\begin{aligned} \mu((E_1 \triangle E_2) \cap B(x, r)) &= \mu((E_1 \setminus E_2) \cap B(x, r)) + \mu((E_2 \setminus E_1) \cap B(x, r)) \\ &= \mu((E_1 \setminus (E_1 \cap E_2)) \cap B(x, r)) + \mu((E_2 \setminus (E_1 \cap E_2)) \cap B(x, r)) \\ &= o(\mu(B(x, r))) \quad \text{as } r \rightarrow 0. \end{aligned}$$

Then assume that $x \in \partial^*(E_1 \cap E_2^c)$ instead. Now we calculate exactly as above, with E_2 replaced by E_2^c . This gives the result. \square

To see that strong locality is stronger than locality, we note that if $E_1, E_2 \subset X$ are sets of locally finite perimeter, then at \mathcal{H} -a.e. $x \in \partial^*E_1 \cap \partial^*E_2$ where (4.8) is satisfied, we have $\theta_{E_1}(x) = \theta_{E_2}(x)$ [3, Proposition 6.2].

Let $\Omega \subset X$ be a set of locally finite perimeter, and let $u \in \text{BV}(\Omega^*)$, with Ω^* open. If the space satisfies the locality condition, we can present the decomposition (2.6) with $A = \partial^*\Omega$ in the simpler form

$$\|Du\|(\Omega^*) = \|Du\|(\Omega^* \setminus \partial^*\Omega) + \int_{\Omega^* \cap \partial^*\Omega} (u^\vee - u^\wedge) \theta_\Omega d\mathcal{H}, \quad (4.9)$$

see [12, Lemma 5.10].

Now, consider a space that does not satisfy the strong locality condition, i.e. there are sets of locally finite perimeter $E_1 \subset E_2 \subset X$ and a set $A \subset \partial^*E_1 \cap \partial^*E_2$ with $\mathcal{H}(A) > 0$, such that

$$\limsup_{r \rightarrow 0} \frac{\mu((E_2 \setminus E_1) \cap B(x, r))}{\mu(B(x, r))} > 0 \quad (4.10)$$

for all $x \in A$. Again we note that $E_2 \setminus E_1$ is also a set of locally finite perimeter. For every $x \in A$ we have $x \in \partial^*(E_2 \setminus E_1)$, due to (4.10). By (2.4) we then know that for \mathcal{H} -a.e. $x \in A$, the lower density of $E_2 \setminus E_1$ is at least $\gamma > 0$. Likewise, the lower densities of E_2^c and E_1 are at least γ for \mathcal{H} -a.e. $x \in A$. Since these three sets are pairwise disjoint, we must have $3\gamma \leq 1$.

Thus we have the following example.

Example 4.6. Let X be a space where $\gamma > 1/3$, where γ is given in (2.4). By the above reasoning, X satisfies the strong locality condition. In particular, in (unweighted) Euclidean spaces, the density of a set of locally finite perimeter is known to be exactly $1/2$ at \mathcal{H} -almost every point of its measure theoretic boundary (see e.g. [2, Theorem 3.59]), so the strong locality condition is satisfied.

By introducing weights, we get further examples.

Example 4.7. Let (X, d, μ) be a space that satisfies the strong locality condition, and replace the measure μ with the weighted measure $w d\mu$, where the weight w is a nonnegative μ -measurable function that is locally bounded and locally bounded away from zero. From the definition of the perimeter measure it follows that the sets of locally finite perimeter in this space are the same as in the unweighted space. Then it easily follows that this

weighted space satisfies the strong locality condition as well. In particular, any weighted Euclidean space equipped with the Euclidean distance and a weighted Lebesgue measure, with the weight locally bounded and locally bounded away from zero, satisfies the strong locality condition.

To begin our analysis of traces, in the following theorem we prove the existence of *interior traces*, which are defined similarly to boundary traces.

Theorem 4.8. *Assume that X satisfies the strong locality condition. Let Ω^* be an open set, let $u \in \text{BV}(\Omega^*)$, and let Ω be a set of locally finite perimeter in Ω^* . Then for \mathcal{H} -a.e. $x \in \Omega^* \cap \partial^* \Omega$, we can define the interior traces $\{T_\Omega u(x), T_{X \setminus \Omega} u(x)\} = \{u^\wedge(x), u^\vee(x)\}$, which satisfy*

$$\lim_{r \rightarrow 0} \int_{B(x,r) \cap \Omega} |u - T_\Omega u(x)|^{Q/(Q-1)} d\mu = 0$$

and

$$\lim_{r \rightarrow 0} \int_{B(x,r) \setminus \Omega} |u - T_{X \setminus \Omega} u(x)|^{Q/(Q-1)} d\mu = 0.$$

Proof. We know that \mathcal{H} -a.e. $x \in \Omega^* \setminus S_u$ is a Lebesgue point of u with

$$\lim_{r \rightarrow 0} \int_{B(x,r)} |u - \tilde{u}(x)|^{Q/(Q-1)} d\mu = 0,$$

see [15, Theorem 3.5]. Moreover, \mathcal{H} -a.e. $x \in \Omega^* \cap \partial^* \Omega$ satisfies (2.4) with $E \hookrightarrow \Omega$, so in these points we can define both $T_\Omega u(x)$ and $T_{X \setminus \Omega} u(x)$ simply as the Lebesgue limit $\tilde{u}(x)$. Let us then consider $x \in \partial^* \Omega \cap S_u$, and again we can assume that (2.4) is satisfied at x with $E \hookrightarrow \Omega$. We know that $x \in \partial^* \{u > t\}$ for every $t \in (u^\wedge(x), u^\vee(x))$. Let T be a countable dense subset of \mathbb{R} such that $\{u > t\}$ is of finite perimeter in Ω^* for every $t \in T$. By the strong locality condition (4.8), we now have for \mathcal{H} -a.e. $x \in \partial^* \Omega \cap S_u$ either

$$\lim_{r \rightarrow 0} \frac{\mu(\{u > t\} \triangle \Omega \cap B(x,r))}{\mu(B(x,r))} = 0 \tag{4.11}$$

for every $t \in (u^\wedge(x), u^\vee(x)) \cap T$, or the same with the substitution $\Omega \hookrightarrow \Omega^c$. Note that the fact that $x \in \partial^* \Omega$ rules out the possibility that we could have (4.11) for some values of t , and (4.11) with $\Omega \hookrightarrow \Omega^c$ for other values of t . Assuming (4.11), it clearly holds for *every* $t \in (u^\wedge(x), u^\vee(x))$. Then by Lemma 4.2 we conclude that for \mathcal{H} -a.e. $x \in \partial^* \Omega \cap S_u$ we can define $T_\Omega u(x) := u^\vee(x)$, and similarly we get $T_{X \setminus \Omega} u(x) = u^\wedge(x)$ — if (4.11) holds with Ω^c instead of Ω , these are the other way around. \square

Next we show that in a space that satisfies the strong locality condition, Theorem 4.3 can be presented in a simpler form.

Theorem 4.9. *Assume that X satisfies the strong locality condition. Let $u \in \text{BV}(X)$. Then for \mathcal{H} -a.e. $x \in S_u$, we have for any $t \in (u^\wedge(x), u^\vee(x))$*

$$\lim_{r \rightarrow 0} \int_{B(x,r) \cap \{u > t\}} |u - u^\vee(x)|^{Q/(Q-1)} d\mu = 0$$

and

$$\lim_{r \rightarrow 0} \int_{B(x,r) \cap \{u \leq t\}} |u - u^\wedge(x)|^{Q/(Q-1)} d\mu = 0.$$

Proof. As before, let T be a countable dense subset of \mathbb{R} such that $\{u > t\}$ is of finite perimeter for every $t \in T$, and let $N \subset X$ be defined as in (4.4). Again, we know that for every $x \in S_u$ and every $t \in (u^\wedge(x), u^\vee(x))$, we have $x \in \partial^* \{u > t\}$. Let $D \subset X$ consist of the points $x \in \partial^* \{u > t\}$ for some $t \in T$, such that either interior trace of u at x does not exist. Since the sets $\{u > t\}$, $t \in T$, are of finite perimeter, by Theorem 4.8 we have $\mathcal{H}(D) = 0$. Now, if $x \in S_u \setminus (D \cup N)$, by Theorem 4.8 we have the result for every $t \in (u^\wedge(x), u^\vee(x)) \cap T$. But since $x \notin N$, we have the result for every $t \in (u^\wedge(x), u^\vee(x))$. \square

Theorem 4.9, and to a lesser extent Theorem 4.3, are analogues of classical results on BV functions. To wit, on Euclidean spaces the theorems can be formulated with the level sets $\{u > t\}$ and $\{u \leq t\}$ replaced by halfspaces (see [9, Section 4.5.9] or [8, p. 213]), but in the case of a metric space with the strong locality condition, we must use the level sets which do not necessarily even have density 1/2 at x , see [15, Example 3.3]. However, the lower and upper densities of these sets are restricted by (2.4).

Having established the existence of interior traces, we proceed to construct boundary traces. However, for our construction to work, we will again need to assume an extension property. First we present two propositions that are similar to [12, Proposition 5.11] and [12, Proposition 5.12], and are originally based on [2, Theorem 3.84] and [2, Theorem 3.86].

Proposition 4.10. *Assume that X satisfies the strong locality condition. Let Ω^* be an open set, and let Ω be a μ -measurable set with $P(\Omega, \Omega^*) < \infty$. Let $u, v \in \text{BV}(\Omega^*)$ and $w := u\chi_{\Omega^* \cap \Omega} + v\chi_{\Omega^* \setminus \Omega}$. Then $w \in \text{BV}(\Omega^*)$ if and only if*

$$\int_{\Omega^* \cap \partial^* \Omega} |T_\Omega u - T_{X \setminus \Omega} v| d\mathcal{H} < \infty. \quad (4.12)$$

Furthermore, we then have

$$\|Dw\|(\Omega^*) = \|Du\|(\Omega^* \cap I) + \|Dv\|(\Omega^* \cap O) + \int_{\Omega^* \cap \partial^* \Omega} |T_\Omega u - T_{X \setminus \Omega} v| \theta_\Omega d\mathcal{H},$$

where I and O are the measure theoretic interior and exterior (points of density one and zero) of Ω .

Proof. Instead of giving the whole proof, we refer to [12, Proposition 5.11]. Crucial in the proof is the existence of interior traces on $\partial^* \Omega$, which is guaranteed by Theorem 4.8 and the finite perimeter of Ω in Ω^* . To obtain the final equality, we also need (4.9). Additionally, in the proof we need the fact that if $w \in \text{BV}(\Omega^*)$, then

$$\|Dw\|(\Omega^* \cap I) = \|Du\|(\Omega^* \cap I) \quad \text{and} \quad \|Dw\|(\Omega^* \cap O) = \|Dv\|(\Omega^* \cap O).$$

The above can be proved with the help of the coarea formula (2.2) as follows. We have

$$\begin{aligned} \|Dw\|(\Omega^* \cap I) &\leq \|Du\|(\Omega^* \cap I) + \|D(w - u)\|(\Omega^* \cap I) \\ &= \|Du\|(\Omega^* \cap I) + \int_{-\infty}^{\infty} P(\{w - u > t\}, \Omega^* \cap I) dt. \end{aligned} \quad (4.13)$$

Here we have $w - u = 0$ μ -almost everywhere in $\Omega^* \cap I$. The inequality opposite to (4.13) is obtained similarly, so we only need to prove that $P(\{w - u > t\}, \Omega^* \cap I) = 0$ for a.e. $t \in \mathbb{R}$. Consider $x \in \Omega^* \cap I$ and $t \in \mathbb{R}$. We have

$$\begin{aligned} &\lim_{r \rightarrow 0} \frac{\mu(B(x, r) \cap \{w - u > t\})}{\mu(B(x, r))} \\ &= \lim_{r \rightarrow 0} \frac{\mu(B(x, r) \cap \{w - u > t\} \cap I)}{\mu(B(x, r))} + \lim_{r \rightarrow 0} \frac{\mu(B(x, r) \cap \{w - u > t\} \setminus I)}{\mu(B(x, r))} \\ &= \lim_{r \rightarrow 0} \frac{\mu(B(x, r) \cap \{w - u > t\} \cap I)}{\mu(B(x, r) \cap I)} + 0 \\ &= \begin{cases} 0 & \text{if } t \geq 0 \\ 1 & \text{if } t < 0. \end{cases} \end{aligned}$$

Thus we have $\partial^* \{w - u > t\} \cap \Omega^* \cap I = \emptyset$ for all $t \in \mathbb{R}$, and it follows that $P(\{w - u > t\}, \Omega^* \cap I) = 0$ for a.e. $t \in \mathbb{R}$ by (2.5). \square

The following proposition on the integrability of traces, or more precisely integrability of lower and upper approximate limits, can be taken directly from [12, Proposition 5.12], where a proof is also given.

Proposition 4.11. *Let $\Omega^* \subset X$ be open, let $u \in \text{BV}(\Omega^*)$, and let $A \subset \Omega^*$ be a bounded Borel set that satisfies $\text{dist}(A, X \setminus \Omega^*) > 0$ and*

$$\mathcal{H}(A \cap B(x, r)) \leq c_A \frac{\mu(B(x, r))}{r} \quad (4.14)$$

for every $x \in A$ and $r \in (0, R]$, where $R \in (0, \text{dist}(A, X \setminus \Omega^))$ and $c_A > 0$ are constants. Then*

$$\int_A (|u^\wedge| + |u^\vee|) d\mathcal{H} \leq C \|u\|_{\text{BV}(\Omega^*)}, \quad (4.15)$$

where $C = C(c_d, c_P, \lambda, A, R, c_A)$.

Now we get the following boundary trace theorem. Here we need an extension property as given in Definition 3.3, but we do not need to require that $\|D(Eu)\|(\partial\Omega) = 0$.

Theorem 4.12. *Assume that X satisfies the strong locality condition. Let Ω be a BV extension domain with constant $c_\Omega > 0$, as well as a set of finite perimeter, and let $u \in \text{BV}(\Omega)$. Then for \mathcal{H} -a.e. $x \in \partial^*\Omega$ we can define the boundary trace $T_\Omega u(x)$ that satisfies*

$$\lim_{r \rightarrow 0} \int_{B(x, r) \cap \Omega} |u - T_\Omega u(x)|^{Q/(Q-1)} d\mu = 0.$$

Moreover, if the assumptions of Proposition 4.11 are satisfied with $\Omega^ = X$, $A = \partial^*\Omega$, and constants $R, c_{\partial^*\Omega} > 0$, then we have*

$$\|T_\Omega u\|_{L^1(\partial^*\Omega, \mathcal{H})} \leq C \|u\|_{\text{BV}(\Omega)},$$

and furthermore $u\chi_\Omega \in \text{BV}(X)$ with $\|u\chi_\Omega\|_{\text{BV}(X)} \leq C \|u\|_{\text{BV}(\Omega)}$ (here $u\chi_\Omega$ naturally just means the zero extension of u to the whole space X). The constant C depends only on $c_d, c_P, \lambda, \Omega, R, c_{\partial^\Omega}$, and c_Ω .*

Proof. Extend u to $Eu \in \text{BV}(X)$. According to Theorem 4.8, for \mathcal{H} -a.e. $x \in \partial^*\Omega$ there exists an interior trace $T_\Omega(Eu)(x)$ that satisfies

$$\begin{aligned} \lim_{r \rightarrow 0} \int_{B(x, r) \cap \Omega} |u - T_\Omega(Eu)(x)|^{Q/(Q-1)} d\mu \\ = \lim_{r \rightarrow 0} \int_{B(x, r) \cap \Omega} |Eu - T_\Omega(Eu)(x)|^{Q/(Q-1)} d\mu = 0, \end{aligned}$$

and so we can define the boundary trace $T_\Omega u(x)$ simply as $T_\Omega(Eu)(x)$ whenever the latter is defined.

To prove the estimate $\|T_\Omega u\|_{L^1(\partial^*\Omega, \mathcal{H})} \leq C\|u\|_{\text{BV}(\Omega)}$, we note that for \mathcal{H} -a.e. $x \in \partial^*\Omega$ we have $T_\Omega u(x) = T_\Omega Eu(x) \in \{(Eu)^\wedge(x), (Eu)^\vee(x)\}$ by Theorem 4.8. By Proposition 4.11 and the definition of an extension domain, we get

$$\begin{aligned} \|T_\Omega u\|_{L^1(\partial^*\Omega, \mathcal{H})} &\leq \| |(Eu)^\wedge| + |(Eu)^\vee| \|_{L^1(\partial^*\Omega, \mathcal{H})} \leq C\|Eu\|_{\text{BV}(X)} \\ &\leq Cc_\Omega\|u\|_{\text{BV}(\Omega)}, \end{aligned} \quad (4.16)$$

with $C = C(c_d, c_P, \lambda, \Omega, R, c_{\partial^*\Omega})$. Finally, by using the fact that $T_\Omega Eu = T_\Omega u$ as well as (4.16), we get

$$\int_{\partial^*\Omega} |T_\Omega(Eu) - T_{X \setminus \Omega} 0| d\mathcal{H} = \int_{\partial^*\Omega} |T_\Omega u| d\mathcal{H} \leq C\|u\|_{\text{BV}(\Omega)} < \infty.$$

By Proposition 4.10, this implies that

$$u\chi_\Omega = (Eu)\chi_\Omega + 0\chi_{X \setminus \Omega} \in \text{BV}(X),$$

with

$$\|D(u\chi_\Omega)\|(X) = \|Du\|(\Omega) + \int_{\partial^*\Omega} |T_\Omega u| \theta_\Omega d\mathcal{H} \leq C\|u\|_{\text{BV}(\Omega)},$$

where the inequality follows from (4.16). \square

Again, we could give essentially the same result and proof with the assumptions used in [12] — instead of assuming strong locality and that Ω is a BV extension domain, we could assume that Ω is a strong BV extension domain, and that both Ω and its complement satisfy the weak measure density condition (2.7). With either set of assumptions, we conclude that any $u \in \text{BV}(\Omega)$ can be extended to the whole space simply by zero extension. This is, of course, in stark contrast with many other classes of functions, such as Newtonian functions. We also see that while the extendability of a BV function can be used to prove the existence of boundary traces, the integrability of the boundary trace $T_\Omega u$ on the boundary $\partial\Omega$ with respect to the measure \mathcal{H} enables, in turn, the function u to be extended by zero extension. This demonstrates the interrelatedness of BV extensions and boundary traces.

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